CHAPTER 9
MATRICES AND DETERMINANTS

While matrix theory, developed in 1858, has many diverse applications, we will direct our discussion toward the objective of solving systems of linear equations.

TERMINOLOGY

We define a matrix as any rectangular array of numbers. We can consider the entries in a table of trigonometric functions as forming a matrix. Also, the entries in a magic square form a matrix. Examples of matrices may be formed from the coefficients and constants of a system of linear equations; that is,

$$2x - 4y = 7$$
$$3x + y = 16$$

can be written

$$\begin{bmatrix} 2 & -4 & 7 \\ 3 & 1 & 16 \end{bmatrix}$$

Notice that we use brackets to enclose the matrix. We could also use double lines; that is,

$$\begin{Vmatrix} 2 & -4 & 7 \\ 3 & 1 & 16 \end{Vmatrix}$$

The numbers used in the matrix are called elements. In the example given we have three columns and two rows. The number of rows and columns are used to determine the dimensions of the matrix. In our example the dimensions of the matrix is 2 x 3. In general, the dimensions of a matrix which has m rows and n columns is called an m x n matrix.

There may occur a matrix with only a row or column in which case it is called either a row or a column matrix. A matrix which has the same number of rows as columns is called a square matrix. Examples of matrices and their dimensions are as follows:

$$\begin{bmatrix} 1 & 7 & 6 \\ 2 & 4 & 8 \end{bmatrix} \quad 2 \times 3$$

$$\begin{bmatrix} 1 & 7 \\ 6 & 2 \\ 3 & 5 \end{bmatrix} \quad 3 \times 2$$

$$\begin{Vmatrix} 2 & 1 \\ 7 & 6 \end{Vmatrix} \quad 2 \times 2 \text{ or square}$$

$$\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \quad 3 \times 1 \text{ or column}$$

$$\begin{bmatrix} 3 & 2 & 1 \end{bmatrix} \quad 1 \times 3 \text{ or row}$$

We will use capital letters, as we did with sets, to describe matrices. We will also include subscripts to give the dimensions; that is,

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 7 & 6 & 5 \end{bmatrix}$$

is the matrix designated by A_{2x3}.

If the situation arises where all of the entries of a matrix are zeros, we call this a zero matrix. The letter we use for a zero matrix is 0. We also include the dimensions; that is, the matrix

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

has the designation 0_{3x2}.

We state that two matrices are equal if and only if they have the same dimensions and their corresponding elements are equal. The elements may have a different appearance such as

$$\begin{bmatrix} 0 & 1 \\ 2 & 4 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & \frac{1}{1} \\ \frac{6}{3} & 4 \end{bmatrix}$$

but the matrices are equal.

Following are examples of matrices which are equal and matrices which are not equal:

$$\begin{bmatrix} 0 & 2 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 4 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 \\ 4 & 7 \end{bmatrix} = \begin{bmatrix} \frac{2}{2} & -\frac{6}{2} \\ 4 & 7 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 7 & 9 \end{bmatrix} \neq \begin{bmatrix} 1 & 2 & 3 \\ 5 & 7 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \neq \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$$

If we interchange rows and columns of a matrix, we form what is called the transpose of the original matrix. We designate the transpose of matrix B as B^T; that is, if

$$B_{2x3} = \begin{bmatrix} 3 & 4 & 7 \\ 5 & 6 & 9 \end{bmatrix}$$

then

$$B^T = \begin{bmatrix} 3 & 5 \\ 4 & 6 \\ 7 & 9 \end{bmatrix}$$

PROBLEMS: Give the dimensions of the following matrices.

1. $\begin{bmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 1 \\ 3 & 2 \\ 2 & 3 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$

ANSWERS:

1. 2x3

2. 3x2

3. 2x2 (square)

PROBLEMS: Give the dimensions of the transpose of the previous problem matrices.

ANSWERS:

1. 3x2

2. 2x3

3. 2x2 (square)

Since two matrices are equal if they have the same corresponding elements, we may find an unknown element of one matrix if we know the elements of an equal matrix; that is, if

$$\begin{bmatrix} 0 & 3 & 2 \\ 1 & 7 & 9 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 2 \\ x & 7 & 9 \end{bmatrix}$$

then $x = 1$

PROBLEMS: Find the unknown elements in the following equal matrices.

1. $\begin{bmatrix} x & 2 & 3 \\ 7 & 9 & y \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 7 & 9 & 10 \end{bmatrix}$

2. $\begin{bmatrix} 1 & x \\ y & 2 \\ 7 & 11 \end{bmatrix} = \begin{bmatrix} z & 6 \\ 5 & 2 \\ 7 & 11 \end{bmatrix}$

3. $\begin{bmatrix} 0 & x \\ y & z \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 4 & 7 \\ 2 & 3 \end{bmatrix}$

ANSWERS:

1. $x = 1$
   $y = 10$

2. $x = 6$
   $y = 5$
   $z = 1$

3. $x = 1$
   $y = 4$
   $z = 7$

PROBLEMS: Write the transpose of the following matrices.

1. $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$

2. $\begin{bmatrix} 3 & 7 & x \\ 2 & 9 & 11 \end{bmatrix}$

3. $\begin{bmatrix} x & y \\ z & w \end{bmatrix}$

ANSWERS:

1. $[0 \ 1 \ 2]$

2. $\begin{bmatrix} 3 & 2 \\ 7 & 9 \\ x & 11 \end{bmatrix}$

3. $\begin{bmatrix} x & z \\ y & w \end{bmatrix}$

ADDITION AND SCALAR MULTIPLICATION

We may add only matrices which have the same dimensions. To add matrices we add the corresponding elements and form the sum as a matrix of the same dimension as those added.

EXAMPLE: Add the matrices A and B if

$$A = \begin{bmatrix} 6 & 2 & 7 \\ -1 & 3 & 0 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 2 & 1 & 3 \\ 0 & -3 & 6 \end{bmatrix}$$

SOLUTION: Write

$$\begin{bmatrix} 6 & 2 & 7 \\ -1 & 3 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 3 \\ 0 & -3 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 6+2 & 2+1 & 7+3 \\ -1+0 & 3-3 & 0+6 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 3 & 10 \\ -1 & 0 & 6 \end{bmatrix}$$

When we add the zero matrix to any matrix, we find the zero matrix is the identity element for addition; that is,

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0 & 2+0 \\ 3+0 & 4+0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Also, in addition of numbers we know that a number plus its negative (additive inverse) equals zero; that is,

$$(3) + (-3) = 0$$

This also holds for matrix addition. To form the negative (additive inverse) of a matrix, we write the matrix with the sign of each element changed; that is, if

$$A_{2x3} = \begin{bmatrix} 1 & 3 & 5 \\ -2 & 6 & 7 \end{bmatrix}$$

then its additive inverse is

$$-A_{2x3} = \begin{bmatrix} -1 & -3 & -5 \\ 2 & -6 & -7 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 3 & 5 \\ -2 & 6 & 7 \end{bmatrix} + \begin{bmatrix} -1 & -3 & -5 \\ +2 & -6 & -7 \end{bmatrix}$$

$$= \begin{bmatrix} 1-1 & 3-3 & 5-5 \\ -2+2 & 6-6 & 7-7 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

By subtraction of matrices we mean the addition of the additive inverse of the subtrahend; that is,

$$A_{2x2} - B_{2x2}$$

is the same as

$$A_{2x2} + (-B_{2x2})$$

EXAMPLE: Subtract B_{3x2} from A_{3x2} if

$$A = \begin{bmatrix} 1 & 2 \\ 5 & 6 \\ 7 & 3 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 0 & 1 \\ 2 & 7 \\ 6 & 3 \end{bmatrix}$$

SOLUTION: Write

$$B = \begin{bmatrix} 0 & 1 \\ 2 & 7 \\ 6 & 3 \end{bmatrix}$$

and

$$-B = \begin{bmatrix} -0 & -1 \\ -2 & -7 \\ -6 & -3 \end{bmatrix}$$

therefore

$$A - B = A + (-B)$$

then

$$\begin{bmatrix} 1 & 2 \\ 5 & 6 \\ 7 & 3 \end{bmatrix} + \begin{bmatrix} -0 & -1 \\ -2 & -7 \\ -6 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 - 0 & 2 - 1 \\ 5 - 2 & 6 - 7 \\ 7 - 6 & 3 - 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 3 & -1 \\ 1 & 0 \end{bmatrix}$$

PROBLEMS: Carry out the indicated operations.

1. $\begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix}$

2. $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 7 \\ 6 \\ 1 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 3 & 7 \end{bmatrix} - \begin{bmatrix} 2 & 3 & 2 \end{bmatrix}$

4. $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix}$

ANSWERS:

1. $\begin{bmatrix} 4 & 3 \\ 3 & -3 \end{bmatrix}$

2. $\begin{bmatrix} 10 \\ 8 \\ 2 \end{bmatrix}$

3. $\begin{bmatrix} -1 & 0 & 5 \end{bmatrix}$

4. $\begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix}$

When solving an equation, in algebra, we isolate the unknown and combine the remainder of the equation; that is, to find the value of x in

$$x + 3 = 7$$

we add the additive inverse of three to each side of the equation to find

$$x + 3 + (-3) = 7 + (-3)$$

and

$$x = 7 + (-3)$$

$$x = 4$$

In dealing with matrices we use the same approach; that is, to solve for the variable matrix in

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 2 & 1 \end{bmatrix}$$

we first add the additive inverse of

$$\begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$$

to each side to find

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} -3 & -2 \\ +1 & 0 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} -3 & -2 \\ +1 & 0 \end{bmatrix}$$

then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 3 & 1 \end{bmatrix}$$

There are two types of multiplication when dealing with matrices. The first is multiplication of a matrix by a constant (scalar). The other is the multiplication of one matrix by another matrix.

When multiplying a matrix by a scalar, we write

scalar K times matrix A

where

$$K = 3$$

and

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 7 \end{bmatrix}$$

is

$$K\begin{bmatrix} 2 & 3 \\ 1 & 7 \end{bmatrix}$$

Every element of A is multiplied by K such that

$$K\begin{bmatrix} 2 & 3 \\ 1 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} K2 & K3 \\ K1 & K7 \end{bmatrix}$$

and K = 3; therefore

$$\begin{bmatrix} K2 & K3 \\ K1 & K7 \end{bmatrix} = \begin{bmatrix} 6 & 9 \\ 3 & 21 \end{bmatrix}$$

PROBLEMS: Multiply each matrix by the given scalar.

1. $\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$, K = 7

2. $[1,\ 7,\ x]$, K = 2

3. $\begin{bmatrix} 0 & 2 & 3 \\ 1 & 3 & 7 \end{bmatrix}$, K = 6

4. $6\begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$

ANSWERS:

1. $\begin{bmatrix} 21 \\ 7 \\ 0 \end{bmatrix}$

2. $[2\ \ 14\ \ 2x]$

3. $\begin{bmatrix} 0 & 12 & 18 \\ 6 & 18 & 42 \end{bmatrix}$

4. $\begin{bmatrix} 12 & 6 \\ 18 & 6 \end{bmatrix}$

In order to explain the multiplication of one matrix by another matrix we use the example

$$[a\ \ b]\begin{bmatrix} x \\ y \end{bmatrix}$$

and state that the product is

$$ax + by$$

Another example is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} aw + by & ax + bz \\ cw + dy & cx + dz \end{bmatrix}$$

The element aw + by in the product matrix is found by multiplying each element in the first row of the first matrix by the corresponding element in the first column of the second matrix.

The element in the second row and first column of the product matrix is found by multiplying each element in the second row of the first matrix by the corresponding element in the first column of the second matrix.

The following examples should clarify matrix multiplication.

EXAMPLE: Multiply

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

SOLUTION: Write

$$\begin{bmatrix} 1a + 2c & 1b + 2d \\ 3a + 4c & 3b + 4d \end{bmatrix}$$

EXAMPLE: Multiply

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 & 5 \\ 0 & 6 \end{bmatrix}$$

SOLUTION: Write

$$\begin{bmatrix} (1 \times 3) + (2 \times 0) & (1 \times 5) + (2 \times 6) \\ (3 \times 3) + (4 \times 0) & (3 \times 5) + (4 \times 6) \end{bmatrix}$$

$$= \begin{bmatrix} 3 + 0 & 5 + 12 \\ 9 + 0 & 15 + 24 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 17 \\ 9 & 39 \end{bmatrix}$$

EXAMPLE: Multiply

$$[1 \;\; 2 \;\; 3] \cdot \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}$$

SOLUTION: Write

$$[(1 \times 3) + (2 \times 4) + (3 \times 0)]$$

$$= [3 + 8 + 0]$$

$$= [11]$$

If two matrices are to be multiplied together, each row in the first matrix must have the same number of elements as each column of the second matrix.

If the left matrix is an $n \times 3$ matrix, the right matrix must be a $3 \times m$. The product matrix will then be an $n \times m$ matrix.

It should be noted that generally matrix multiplication is not commutative. This is shown by the following:

EXAMPLE: Multiply

$$\begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$$

and

$$\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}$$

SOLUTION: Write

$$\begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} (3 \times 2) + (2 \times 1) & (3 \times 1) + (2 \times 4) \\ (1 \times 2) + (0 \times 1) & (1 \times 1) + (0 \times 4) \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 11 \\ 2 & 1 \end{bmatrix}$$

and then write

$$\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} (2 \times 3) + (1 \times 1) & (2 \times 2) + (1 \times 0) \\ (1 \times 3) + (4 \times 1) & (1 \times 2) + (4 \times 0) \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 4 \\ 7 & 2 \end{bmatrix}$$

which is different from the first product. Since multiplication of matrices is not commutative, we must define multiplication as being either right or left multiplication; that is, xy means left multiplication of y by x, and it also means right multiplication of x by y. Therefore, we find if we are to multiply x by y we have two products to choose from; that is, if

$$x = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

and

$$y = \begin{bmatrix} 3 & 2 \\ 1 & 6 \end{bmatrix}$$

then·

$$xy = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 \\ 1 & 6 \end{bmatrix} = \begin{bmatrix} 5 & 14 \\ 13 & 30 \end{bmatrix}$$

and

$$yx = \begin{bmatrix} 3 & 2 \\ 1 & 6 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 9 & 14 \\ 19 & 26 \end{bmatrix}$$

The identity matrices for multiplication are those square matrices which have the elements which form the diagonal from upper left to lower right equal 1 while all other entries are equal to 0; that is, the matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is the 2x2 identity matrix for multiplication. If we multiply

$$\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

we find the product to be

$$\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$$

PROBLEMS: Multiply

1. $[1 \; 3 \; 7] \cdot \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$

2. $\begin{bmatrix} 3 & -2 \\ 1 & -6 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$

3. $\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

4. $\begin{bmatrix} -2 & 1 \\ -4 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$

ANSWERS:

1. $[18]$

2. $\begin{bmatrix} 0 & -1 \\ -16 & -11 \end{bmatrix}$

3. $\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$

4. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Notice that problem number four infers that when xy equals zero, it is not necessary for either x or y to be zero.

We also make the statement that the distributive laws

$$Ax + Ay = A(x + y)$$

and

$$xA + yA = (x + y)A$$

hold as does the associative law

$$A(BC) = (AB)C$$

DETERMINANT FUNCTION

We may evaluate a square 2x2 matrix and associate the matrix with a real number by adding the product of the elements on one diagonal to the negative of the product of the elements on the other diagonal; that is,

$$\begin{bmatrix} a & B \\ c & D \end{bmatrix}$$

may be associated with aD - Bc. We call this number the determinant of the matrix. NOTE: This procedure applies to second-order determinants only. The determinant function of matrix A is given by $\delta(A)$.

EXAMPLE: If

$$A = \begin{bmatrix} -5 & 1 \\ 3 & 2 \end{bmatrix}$$

find $\delta(A)$

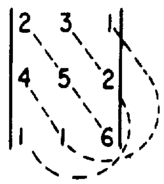
SOLUTION: Write

$$\delta(A) = (-5 \times 2) - (3 \times 1)$$
$$= -10 - 3$$
$$= -13$$

We may find the determinant of any square matrix of any dimension. If we have

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 4 & 5 & 2 \\ 1 & 1 & 6 \end{bmatrix}$$

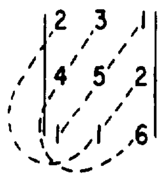
then to find $\delta(A)$ we follow the pattern of



to find the first set of diagonals and write

$$2 \cdot 5 \cdot 6 + 4 \cdot 1 \cdot 1 + 1 \cdot 2 \cdot 3$$

We next follow the pattern of



to find the set of diagonals which are to be subtracted from the first set. We write

$$1 \cdot 5 \cdot 1 + 3 \cdot 4 \cdot 6 + 2 \cdot 1 \cdot 2$$

and then

$$(2 \cdot 5 \cdot 6 + 4 \cdot 1 \cdot 1 + 1 \cdot 2 \cdot 3)$$
$$- (1 \cdot 5 \cdot 1 + 3 \cdot 4 \cdot 6 + 2 \cdot 1 \cdot 2)$$
$$= (60 + 4 + 6) - (5 + 72 + 4)$$
$$= 70 - 81$$
$$= -11$$

which is the determinant of A. Therefore, if

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 4 & 5 & 2 \\ 1 & 1 & 6 \end{bmatrix}$$

then

$$\delta(A) = -11$$

NOTE: This pattern applies to third-order determinants only.

In some cases we do not evaluate a matrix to find the determinant but merely write the matrix elements and enclose them by vertical bars; that is, if

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

then

$$\delta(A) = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

The order of the determinant is determined by the number of elements in any row or column. In this case the order of the determinant is three.

In the preceding example we may write $\delta(A)$ in the form of

$$(aei + bfg + cdh) - (ceg + bdi + afh)$$

PROBLEMS: Find the determinants of the following matrices.

1. $\begin{bmatrix} 3 & 2 \\ -1 & 2 \end{bmatrix}$

2. $\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 3 & 2 \\ 1 & 1 & 4 \\ 1 & 1 & 2 \end{bmatrix}$

ANSWERS:

1. 8

2. -2

3. 4

INVERSE OF A MATRIX

When we multiply two matrices together and find that the product is the multiplicative identity, we say that one of the matrices is the inverse of the other. This is similar to arithmetic in which

$$\frac{1}{a} \cdot \frac{a}{1} = 1$$

and we find $\frac{1}{a}$ is the inverse of a. Notice that this holds as long as "a" does not equal zero. This same requirement is made with matrices; that is, a matrix has an inverse as long as the determinant of the matrix is not equal to zero.

If we multiply the matrices A and B where

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

we find

$$\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

therefore the matrix A is the inverse of matrix B and matrix B is the inverse of matrix A.

Generally, to designate an inverse of a matrix we write, if

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then the inverse of A is

$$A^{-1} = \frac{1}{\delta(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The reason for multiplying by $\frac{1}{\delta(A)}$ is shown by the following example.

EXAMPLE: Find the inverse of matrix A if

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$$

SOLUTION: Interchange the 1 and 2 and then change the signs of 3 and 4. Write

$$\begin{bmatrix} 1 & -3 \\ -4 & 2 \end{bmatrix}$$

If we now multiply

$$\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -3 \\ -4 & 2 \end{bmatrix}$$

we find the product to be

$$\begin{bmatrix} 2 - 12 & -6 + 6 \\ 4 - 4 & -12 + 2 \end{bmatrix}$$

$$= \begin{bmatrix} -10 & 0 \\ 0 & -10 \end{bmatrix}$$

In order to make this product equal the multiplicative inverse, we multiply by $\frac{1}{-10}$ which gives

$$\frac{1}{-10} \begin{bmatrix} -10 & 0 \\ 0 & -10 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-10}{-10} & 0 \\ 0 & \frac{-10}{-10} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Also notice that the determinant $\delta(A)$ of

$$\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$$

is

$$(2) - (12) = -10$$

We may now write the general formula for the inverse of a matrix A as follows. If

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then

$$A^{-1} = \frac{1}{\delta(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

EXAMPLE: Find the inverse of matrix A if

$$A = \begin{bmatrix} 5 & 6 \\ 4 & 3 \end{bmatrix}$$

SOLUTION: Write

$$A^{-1} = \frac{1}{\delta(A)} \begin{bmatrix} 3 & -6 \\ -4 & 5 \end{bmatrix}$$

and

$$\delta(A) = 15 - 24$$
$$= -9$$

Then

$$A^{-1} = \frac{1}{-9} \begin{bmatrix} 3 & -6 \\ -4 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{3}{9} & \frac{2}{3} \\ \frac{4}{9} & -\frac{5}{9} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{4}{9} & -\frac{5}{9} \end{bmatrix}$$

To verify this we write

$$\begin{bmatrix} 5 & 6 \\ 4 & 3 \end{bmatrix} \cdot \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{4}{9} & -\frac{5}{9} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{5}{3} + \frac{24}{9} & \frac{10}{3} - \frac{30}{9} \\ -\frac{4}{3} + \frac{12}{9} & \frac{8}{3} - \frac{15}{9} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

PROBLEMS: Find the inverse of the following matrices.

1. $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$

3. $\begin{bmatrix} 4 & 5 \\ 0 & 1 \end{bmatrix}$

ANSWERS:

1. $\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

2. $\begin{bmatrix} \frac{3}{5} & -\frac{2}{5} \\ \frac{1}{5} & \frac{1}{5} \end{bmatrix}$

3. $\begin{bmatrix} \frac{1}{4} & -\frac{5}{4} \\ 0 & 1 \end{bmatrix}$

DETERMINANTS

We have previously determined that the difference between a matrix and a determinant is that a matrix is an array of numbers and a determinant represents a particular number. When we found the particular number a determinant represented, we called this operation "expanding the determinant"; that is, to expand the determinant

$$\begin{vmatrix} 1 & 3 & 1 \\ 4 & 1 & 2 \\ 5 & 6 & 3 \end{vmatrix}$$

we write

$$(1 \cdot 1 \cdot 3 + 4 \cdot 6 \cdot 1 + 3 \cdot 2 \cdot 5)$$
$$- (5 \cdot 1 \cdot 1 + 3 \cdot 4 \cdot 3 + 1 \cdot 6 \cdot 2)$$
$$= (3 + 24 + 30) - (5 + 36 + 12)$$
$$= 57 - 53$$
$$= 4$$

EXPANSION BY MINORS

Another way in which a determinant may be expanded is by expansion by minors. If we have the determinant

$$\begin{vmatrix} 1 & 3 & 1 \\ 4 & 1 & 2 \\ 5 & 6 & 3 \end{vmatrix}$$

then the minor of the element 3 in the top row is the determinant resulting from the deletion of both row and column that contains the element 3; that is, the minor of 3 is

$$\begin{vmatrix} 1 & 3 & 1 \\ 4 & 1 & 2 \\ 5 & 6 & 3 \end{vmatrix}$$

or

$$\begin{vmatrix} 4 & 2 \\ 5 & 3 \end{vmatrix}$$

Also, the minor of the element 4 is

$$\begin{vmatrix} 1 & 3 & 1 \\ 4 & 1 & 2 \\ 5 & 6 & 3 \end{vmatrix}$$

or

$$\begin{vmatrix} 3 & 1 \\ 6 & 3 \end{vmatrix}$$

In order to expand the determinant by minors of the first column we write the minors of 1, 4, and 5 which are:

$$\text{the minor of } 1 = \begin{vmatrix} 1 & 2 \\ 6 & 3 \end{vmatrix}$$

$$\text{the minor of } 4 = \begin{vmatrix} 3 & 1 \\ 6 & 3 \end{vmatrix}$$

$$\text{and the minor of } 5 = \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix}$$

We now multiply the element by its minor and by the sign of the element location from the pattern

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

which gives

$$1\begin{vmatrix} 1 & 2 \\ 6 & 3 \end{vmatrix}$$

$$-4\begin{vmatrix} 3 & 1 \\ 6 & 3 \end{vmatrix}$$

and

$$5\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix}$$

which results in

$$1\begin{vmatrix} 1 & 2 \\ 6 & 3 \end{vmatrix} = 1\ (3 - 12)$$
$$= -9$$

and

$$-4\begin{vmatrix} 3 & 1 \\ 6 & 3 \end{vmatrix} = -4\ (9 - 6)$$
$$= -12$$

and

$$5\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} = 5\ (6 - 1)$$
$$= 25$$

which, when added, gives the expansion of the determinant to be

$$- 9 - 12 + 25$$

$$= 4$$

which is the same result as we previously determined.

The expansion of a determinant may be accomplished according to any row or column with the same results. If we expand the determinant

$$\begin{vmatrix} 3 & 2 & 1 \\ 2 & 4 & 3 \\ 1 & 1 & 1 \end{vmatrix}$$

about the first row, we have

$$3\begin{vmatrix} 4 & 3 \\ 1 & 1 \end{vmatrix} -2 \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} +1 \begin{vmatrix} 2 & 4 \\ 1 & 1 \end{vmatrix}$$

$$= 3 + 2 - 2$$

$$= 3$$

If we expand the determinant about the second column, we have

$$-2\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} +4 \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} -1 \begin{vmatrix} 3 & 1 \\ 2 & 3 \end{vmatrix}$$

$$= 2 + 8 - 7$$

$$= 3$$

It should be noted that a fourth-order determinant has minors which are third-order. The third-order determinants may be defined by second-order determinants. Therefore, through the process of expansion by minors, a high order determinant may finally, through many steps, be expressed by second-order determinants.

PROBLEMS: Expand the following determinants by minors about the row or column indicated. Check your solution by expanding about any other row or column.

1. $\begin{vmatrix} 3 & 1 & 2 \\ 4 & 5 & 6 \\ 0 & 1 & 4 \end{vmatrix}$ about row 1.

2. $\begin{vmatrix} -2 & 1 & -3 \\ 2 & 3 & 1 \\ 1 & -1 & 1 \end{vmatrix}$ about row 3.

3. $\begin{vmatrix} 3 & 5 & 2 \\ 4 & 0 & 1 \\ -1 & 2 & -2 \end{vmatrix}$ about column 2.

ANSWERS:

1. 34

2. 6

3. 45

PROPERTIES

The following properties of determinants may be used to simplify the determinants. These properties apply to any order determinants but will be given with examples using third-order determinants.

(1) The determinant is zero if two rows or two columns are identical.

(2) The sign of the value of a determinant is changed if two rows or two columns of the determinant are interchanged.

(3) The value of a determinant is not changed if all rows and columns are interchanged in order.

(4) The value of a determinant is zero if every element in a row or column is zero.

(5) The value of a determinant is increased by the factor K if any row or column is multiplied by K.

(6) The elements of any row or column may be multiplied by a real number K and these products then added to the elements of another row or column respectively without changing the value of the determinant.

Examples for the listed properties are as follows:

(1) $$\begin{vmatrix} 3 & 3 & 1 \\ 2 & 2 & 6 \\ -1 & -1 & 5 \end{vmatrix} = (30 - 18 - 2) - (-2 + 30 - 18)$$

$$= 30 - 18 - 2 + 2 - 30 + 18$$

$$= 0$$

(2)
$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 1 & 5 \\ -1 & 2 & 3 \end{vmatrix} = (3 - 10 + 24) - (-3 + 24 + 10) = 3 - 10 + 24 + 3 - 24 - 10 = -14$$

and

$$\begin{vmatrix} 4 & 1 & 5 \\ 1 & 2 & 3 \\ -1 & 2 & 3 \end{vmatrix} = (24 + 10 - 3) - (-10 + 24 + 3) = 24 + 10 - 3 + 10 - 24 - 3 = 14$$

(3)
$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 1 & 3 \\ 2 & 2 & 4 \end{vmatrix} = (4 + 24 + 12) - (6 + 32 + 6) = 4 + 24 + 12 - 6 - 32 - 6 = -4$$

and

$$\begin{vmatrix} 1 & 4 & 2 \\ 2 & 1 & 2 \\ 3 & 3 & 4 \end{vmatrix} = (4 + 24 + 12) - (6 + 6 + 32) = 4 + 24 + 12 - 6 - 6 - 32 = -4$$

(4)
$$\begin{vmatrix} 4 & 0 & 6 \\ 2 & 0 & 3 \\ 1 & 0 & 5 \end{vmatrix} = (0 + 0 + 0) - (0 + 0 + 0) = 0 - 0 = 0$$

(5)
$$\begin{vmatrix} 2 & 1 & 1 \\ 4 & 5 & 6 \\ 2 & 1 & 3 \end{vmatrix} = (30 + 4 + 12) - (10 + 12 + 12) = 30 + 4 + 12 - 10 - 12 - 12 = 12$$

and if K = 2 then

$$\begin{vmatrix} 4 & 1 & 1 \\ 8 & 5 & 6 \\ 4 & 1 & 3 \end{vmatrix} = (60 + 24 + 8) - (20 + 24 + 24) = 60 + 24 + 8 - 20 - 24 - 24 = 24$$

(6)
$$\begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 3 & 1 & 3 \end{vmatrix} = (12 + 3 + 1) - (6 + 3 + 2) = 12 + 3 + 1 - 6 - 3 - 2 = 5$$

and

$$\begin{vmatrix} 2 \cdot 2 & 1 & 1 \\ 2 \cdot 1 & 2 & 1 \\ 2 \cdot 3 & 1 & 3 \end{vmatrix}$$

then

$$\begin{vmatrix} 2 & 1 & 1+4 \\ 1 & 2 & 1+2 \\ 3 & 1 & 3+6 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 1 & 5 \\ 1 & 2 & 3 \\ 3 & 1 & 9 \end{vmatrix} = (36 + 9 + 5) - (30 + 6 + 9) = 36 + 9 + 5 - 30 - 6 - 9 = 5$$

EXAMPLE: Evaluate

$$\begin{vmatrix} 1 & 3 & 7 \\ 2 & 9 & 21 \\ -3 & 20 & 16 \end{vmatrix}$$

SOLUTION: It is obvious that if we expand by minors we will encounter large numbers; therefore, we will use some of the properties of determinants.

(1) Multiply the first row by -2 and add product to the second row to find

$$\begin{vmatrix} 1 & 3 & 7 \\ 2 - 2 & 9 - 6 & 21 - 14 \\ -3 & 20 & 16 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 3 & 7 \\ 0 & 3 & 7 \\ -3 & 20 & 16 \end{vmatrix}$$

then multiply the first row by 3 and add product to the third row to find

$$\begin{vmatrix} 1 & 3 & 7 \\ 0 & 3 & 7 \\ -3+3 & 20+9 & 16+21 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 3 & 7 \\ 0 & 3 & 7 \\ 0 & 29 & 37 \end{vmatrix}$$

Now we expand by minors about the first column; that is,

$$\begin{vmatrix} 1 & 3 & 7 \\ 0 & 3 & 7 \\ 0 & 29 & 37 \end{vmatrix}$$

$$= 1\begin{vmatrix} 3 & 7 \\ 29 & 37 \end{vmatrix} - 0\begin{vmatrix} 3 & 7 \\ 29 & 37 \end{vmatrix} + 0\begin{vmatrix} 3 & 7 \\ 3 & 7 \end{vmatrix}$$

$$= 1\begin{vmatrix} 3 & 7 \\ 29 & 37 \end{vmatrix} = 111 - 203$$

$$= -92$$

EXAMPLE: Evaluate

$$\begin{vmatrix} 1 & 3 & -2 \\ 6 & 7 & 21 \\ 1 & 3 & -2 \end{vmatrix}$$

SOLUTION: The first and third rows are identical; therefore, the determinant value is zero.

SOLUTION OF SYSTEMS OF LINEAR EQUATIONS

Second-order determinants may be used to solve systems of two linear equations in two unknowns. If we have two equations such as

$$ax + by = c$$

$$a_1x + b_1y = c_1$$

we may write

$$x = \frac{\begin{vmatrix} c & b \\ c_1 & b_1 \end{vmatrix}}{\begin{vmatrix} a & b \\ a_1 & b_1 \end{vmatrix}}$$

and

$$y = \frac{\begin{vmatrix} a & c \\ a_1 & c_1 \end{vmatrix}}{\begin{vmatrix} a & b \\ a_1 & b_1 \end{vmatrix}}$$

Note that the denominators are the coefficients of x and y in the same arrangement as given in the problem. Also note that the numerator is formed by replacing the column of coefficients of the desired unknown by the column of constants or the right side of the equations.

EXAMPLE: Solve the system

$$4x + 2y = 5$$

$$3x - 4y = 1$$

SOLUTION: Write

$$x = \frac{\begin{vmatrix} 5 & 2 \\ 1 & -4 \end{vmatrix}}{\begin{vmatrix} 4 & 2 \\ 3 & -4 \end{vmatrix}} = \frac{-20 - 2}{-16 - 6} = \frac{-22}{-22} = 1$$

and

$$y = \frac{\begin{vmatrix} 4 & 5 \\ 3 & 1 \end{vmatrix}}{\begin{vmatrix} 4 & 2 \\ 3 & -4 \end{vmatrix}} = \frac{4 - 15}{-16 - 6} = \frac{-11}{-22} = \frac{1}{2}$$

The method of solving the system in this example is called Cramer's rule; that is, when we solve a system of linear equations by the use of determinants, we are using Cramer's rule.

PROBLEMS: Solve for the unknown in the following systems by use of Cramer's rule.

1. $x + 2y = 4$
   $-x + 3y = 1$
2. $3x + 2y = 12$
   $4x + 5y = 2$
3. $x - 2y = -1$
   $2x + 3y = 12$

ANSWERS:

1. $x = 2$
   $y = 1$

2. $x = 8$

   $y = -6$

3. $x = 3$

   $y = 2$

Cramer's rule may be applied to systems of three linear equations in three unknowns. We use the same technique as given in previous examples; that is, if we have

$$ax + by + cz = d$$

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

we may write

$$x = \frac{\begin{vmatrix} d & b & c \\ d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a & b & c \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}$$

$$y = \frac{\begin{vmatrix} a & d & c \\ a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a & b & c \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}$$

$$z = \frac{\begin{vmatrix} a & b & d \\ a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \end{vmatrix}}{\begin{vmatrix} a & b & c \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}$$

EXAMPLE: Use Cramer's rule to solve the system

$$2x + 3y - z = 2$$

$$x - 2y + 2z = -10$$

$$3x + y - 2z = 1$$

SOLUTION: Write

$$x = \frac{\begin{vmatrix} 2 & 3 & -1 \\ -10 & -2 & 2 \\ 1 & 1 & -2 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & -1 \\ 1 & -2 & 2 \\ 3 & 1 & -2 \end{vmatrix}}$$

$$y = \frac{\begin{vmatrix} 2 & 2 & -1 \\ 1 & -10 & 2 \\ 3 & 1 & -2 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & -1 \\ 1 & -2 & 2 \\ 3 & 1 & -2 \end{vmatrix}}$$

and

$$z = \frac{\begin{vmatrix} 2 & 3 & 2 \\ 1 & -2 & -10 \\ 3 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & -1 \\ 1 & -2 & 2 \\ 3 & 1 & -2 \end{vmatrix}}$$

Then, solving for the unknowns find that

$$x = -2$$

$$y = 1$$

$$z = -3$$

It should be noted that when determining either the numerator or the denominator in solving systems similar to the previous example, the properties of determinants should be used when possible.

PROBLEMS: Find the solution to the following systems of linear equations.

1. $x + 2y - 3z = -7$

   $3x - y + 2z = 8$

   $2x - y + z = 5$

2. $2x - 3y - 5z = 5$

$x + y - z = 2$

$x - 2y - 3z = 3$

3. $x - 3y - 3z = -2$

$3x - 2y + 2z = -3$

$2x + y - z = 5$

ANSWERS:

1. $x = 1$

$y = -1$

$z = 2$

2. $x = -1$

$y = 1$

$z = -2$

3. $x = 1$

$y = 2$

$z = -1$

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